PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Recent Advances in the Stability Theory of Toroidal Plasmas

R. J. Hastie

Phil. Trans. R. Soc. Lond. A 1981 **300**, 559-567
doi: 10.1098/rsta.1981.0084

Phil. Trans. R. Soc. Lond. A **300**, 559–567 (1981) [559]
Printed in Great Britain

Recent advances in the stability theory of toroidal plasmas

By R. J. Hastie
Euratom/U.K.A.E.A. Fusion Association, Culham Laboratory, Abingdon, Oxfordshire OX14 3DB, U.K.

Many of the most persistent instabilities of a magnetically confined plasma have short wavelength perpendicular to the magnetic field but long wavelength parallel to it. Such instabilities are difficult to treat in a toroidal system because the simple eikonal representation of short wavelength oscillations

$$X(r) = Y(r)\,e^{iS(r)/\delta}$$

with $\delta \ll 1$ proves to be incompatible with the other requirements of toroidal periodicity and long parallel wavelength (which would require $B\cdot\nabla S = 0$).

A new method of representing perturbations in a torus will be outlined. By using this, the two-dimensional stability problem posed by an axisymmetric toroidal equilibrium can be reduced to that of solving a one-dimensional eigenvalue equation.

This technique essentially completes the linear stability theory of magnetohydrodynamic modes in a toroidal plasma, and is also applicable to the investigation of micro-instabilities that are described by the Vlasov–Maxwell equations.

1. Introduction

The most important magnetic confinement devices currently being developed and under construction are axisymmetric toroidal devices. Of these toroidal pinches provide one example, while tokamaks exemplified by the JET and T.F.T.R. designs dominate toroidal confinement research. Theoretical studies of plasma stability in such devices are of importance as one factor in determining what value of the parameter β ($\equiv 2p/B^2$) can be achieved in a torus before instabilities cause an unacceptable deterioration in confinement. Much effort has been devoted to theoretical stability analyses by means of a magnetohydrodynamic (m.h.d.) plasma model, but although a prescription for completely determining the m.h.d. stability of an infinite *cylindrically symmetric* plasma equilibrium has been available for more than a decade it is only recently that an equivalent method has been developed for the realistic case of *toroidal* plasmas.

This paper is devoted to discussing the additional difficulty presented by the toroidal problem and to outlining the progress in linear stability theory which has now made possible the complete determination of m.h.d. stability thresholds and growth rates in a torus.

Although the main successes have been in the m.h.d. model the new methods are equally applicable to the more detailed kinetic descriptions required in the study of micro-instabilities, and these applications will also be discussed.

2. Axisymmetric toroidal equilibria and coordinate systems

The magnetic field in an axisymmetric toroidal confinement device has components in both the toroidal and poloidal directions so that magnetic lines of force lie on a set of nested toroidal surfaces – magnetic surfaces – enclosing and degenerating into a line, the magnetic axis.

Magnetic surfaces are conveniently labelled by the poloidal flux ψ contained by them, and toroidal and poloidal angles ζ and θ complete the coordinate system.

An important equilibrium parameter is the 'safety factor' $q(\psi)$, so called because of its influence on stability, defined so that $2\pi q$ is the angle of rotation around the symmetry axis performed by any field line in passing once around the magnetic axis. The rate of change of q from surface to surface q' ($\equiv \mathrm{d}q/\mathrm{d}\psi$) is known as magnetic shear and is related to the change of direction of field lines on adjacent magnetic surfaces.

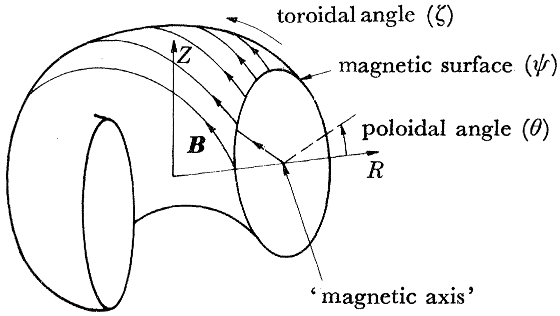

FIGURE 1. Toroidal confinement system and (ψ, θ, ζ)-coordinates.

3. MAGNETOHYDRODYNAMIC STABILITY

(a) *A brief historical survey of theory*

In 1960 Newcomb, analysing the stability of an ideal m.h.d. (i.e. $\eta \equiv 0$) cylindrically symmetric plasma, reduced the problem to that of solving a single second-order ordinary differential equation. Inclusion of resistivity (Furth *et al.* 1963; Coppi *et al.* 1966) revealed new instabilities requiring consideration of an infinitesimal 'resistive layer' in the neighbourhood of the 'resonant surface', i.e. that magnetic surface within the plasma on which $\boldsymbol{k}\cdot\boldsymbol{B} = 0$, with $\boldsymbol{k}$ the wave number of the perturbation. A complete analysis of these resistive instabilities was also possible, by solving a pair of coupled ordinary differential equations. Thus a precise quantitative method for calculating the m.h.d. stability of any given cylindrical equilibrium had been formulated by the mid-sixties.

During the same period investigations of toroidal plasma stability were dominated by the search for analytic stability criteria. An early success came with the derivation by Mercier (1960) of a condition which was necessary but not sufficient to ensure stability. This predicted an upper bound for the β which could be achieved in a torus. It did not however determine the threshold value of β for instability.

Much insight was obtained from a series of theoretical studies (see, for example, Ware & Haas 1966; Frieman *et al.* 1973; Shafranov 1970; Bussac *et al.* 1975) based on an expansion of equilibrium and perturbed quantities in powers of the inverse aspect ratio $\epsilon = a/R =$ (plasma radius)/(major toroidal radius), but many of these calculations were restricted to a low-pressure ordering in which $\beta \sim O(\epsilon^2)$ and to tori of circular cross section.

The 1970s saw the emergence of a new quantitative approach to the toroidal stability problem with the development of large two-dimensional computer codes, either integrating the linearized m.h.d. equations in time (Sykes & Wesson 1974; Schneider & Bateman 1975) or solving the two-dimensional eigenvalue equations (Berger *et al.* 1977; Johnson *et al.* 1977).

Because of the axisymmetry property different Fourier modes $e^{in\zeta}$ are independent so that each toroidal mode number n could be investigated separately. In practice however these codes were restricted by computer limitations to $n \lesssim 5$. But whereas in the cylindrical problem Newcomb had been able to show that of all the Fourier modes $e^{i(m\theta+kz)}$ only $m = 0$ and $m = 1$ need be studied since they were the most unstable, no analogous theorem involving the toroidal mode number n was apparent. Numerical results did not indicate that in practice low values of n were the most unstable for internal pressure-driven modes. On the contrary, there were indications that, as n was increased, greater instability of internal modes was usually encountered. This was the situation when Wesson (1978) gave a comprehensive review of the theory of the m.h.d. stability of tokamak plasmas. An analytic treatment of short-wavelength (high n) modes was still required to complement the computed, low n, results but, as we now discuss, such a theory posed special problems in a torus.

(b) *Short wavelength eigenmodes in a torus*

An important feature of many short-wavelength instabilities in a magnetized plasma is the extreme anisotropy of this wavelength in the directions parallel to and perpendicular to the confining magnetic field. This is true for many of the micro-instabilities that are predicted by the Vlasov–Maxwell plasma equations, where long parallel wavelength is frequently necessitated by the avoidance of ion Landau damping. It is also true for m.h.d. instabilities where long parallel wavelengths minimize distortion of the stabilizing magnetic field which is strongly coupled to the fluid. A central problem of plasma stability theory has therefore been to find a way of representing such anisotropic perturbations in a toroidal system.

It is natural to attempt to describe short-wavelength oscillations in an inhomogeneous medium by means of an eikonal representation of the form

$$\xi = A\, e^{iS/\delta}, \tag{1}$$

where δ is a small parameter representing the ratio of the wavelength of the oscillations to the scale of the system. In the present problem the necessity for long wavelength parallel to $\boldsymbol{B}$ requires that $\boldsymbol{B}\cdot\nabla S = 0$, Fourier decomposition $\sim e^{in\zeta}$ determines the ζ-dependence of S, and the toroidal mode number n naturally fills the role of the parameter δ^{-1} in equation (1), so that taking $\xi = A\, e^{inS}$ one finds that S must take the form

$$S(\theta, \zeta, \psi) = \zeta - q(\psi)\,\theta + \int^{\psi} \theta_0 q'\, d\psi, \tag{2}$$

where θ is the (generalized†) poloidal angle, $\theta_0(\psi)$ may be regarded as a reference angle about which the mode is centred, and the rather curious form taken for the arbitrary function containing $\theta_0(\psi)$ is for later convenience. This eikonal is however clearly unacceptable since

† The angle θ used here is defined by $\theta = 2\pi \int^{l} B_\zeta (RB_\theta)^{-1} dl \Big/ \oint B_\zeta (RB_\theta)^{-1}\, dl$ where l is arc length in the poloidal direction, so that ζ, θ, ψ form a non-orthogonal coordinate system in which the field lines are straight, $d\zeta/d\theta = q(\psi)$. Many authors go further and use the Hamada (1958) coordinate system $(\hat{\zeta}, \hat{\theta}, V(\psi))$ where

$$\hat{\zeta} = \zeta - \int^{l} B_\zeta (RB_\theta)^{-1}\, dl + 2\pi q \int^{l} B_\theta^{-1} dl \Big/ \oint B_\theta^{-1} dl;\ \hat{\theta} = 2\pi \int^{l} B_\theta^{-1} dl \Big/ \oint B_\theta^{-1} dl$$

and $V(\psi)$ is the volume within a magnetic surface. In these non-orthogonal coordinates the field lines are also straight, $d\hat{\zeta}/d\hat{\theta} = q(\psi)$, and in addition the Jacobean of the transformation to $(\hat{\zeta}, \hat{\theta}, V)$, $(\nabla\hat{\zeta}\cdot\nabla\hat{\theta}\wedge\nabla V)^{-1}$, is a constant.

e^{inS} cannot be a periodic function of θ if $q(\psi)$ is not a rational constant. Furthermore aperiodicity of e^{inS} cannot be compensated by an appropriate choice of the amplitude A since A is assumed to be slowly varying. Clearly what is needed is a representation for standing waves in a torus that reconciles the anisotropy of wavelengths parallel to and perpendicular to $\boldsymbol{B}$ with toroidal periodicity, but which still allows one to exploit the small parameter n^{-1}. Recently such a representation (Connor *et al.* 1978, 1979*a*, *b*) has been developed and this has made possible rigorous stability analyses of short-wavelength m.h.d. modes in any axisymmetric toroidal equilibrium.

In essence the method is simple. In any axisymmetric toroidal system the linear stability threshold is determined by a two-dimensional eigenvalue equation:

$$\mathrm{L}(\theta, \psi)\, \xi(\theta, \psi) = \lambda \xi(\theta, \psi). \tag{3}$$

The operator L is periodic in θ and the eigenfunction ξ must also be periodic. In the ideal m.h.d. problem L is a second-order differential, containing the differential operators $\mathrm{L}_{\parallel}$ and $\mathrm{L}_{\perp}$, where

$$\mathrm{L}_{\parallel} = \frac{\partial}{\partial\theta} - inq(\psi); \quad \mathrm{L}_{\perp} = \frac{1}{n}\frac{\partial}{\partial\psi}. \tag{4}$$

Were use of the eikonal representation (equations (1) and (2)) permissible these operators would transform as

$$\mathrm{L}_{\parallel}\xi \rightarrow \mathrm{d}A/\mathrm{d}\theta; \quad \mathrm{L}_{\perp}\xi \rightarrow \mathrm{i}A(\theta_0-\theta)\, q' + O(n^{-1}) \tag{5}$$

and the two-dimensional equation (3) would have been reduced in leading order of an expansion in n^{-1} to a one-dimensional equation for $A(\theta)$ with ψ and $\theta_0(\psi)$ appearing only as parameters.

With this in mind we express ξ (Connor *et al.* 1978) in the form

$$\xi(\theta, \psi) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}m\theta} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}my}\hat{\xi}(y, \psi)\, \mathrm{d}y \tag{6}$$

which ensures that ξ is periodic in θ. The function $\hat{\xi}$ need not be a periodic function of y, but any $\hat{\xi}$ which is a solution of

$$\mathrm{L}(y, \psi)\, \hat{\xi}(y, \psi) = \lambda \hat{\xi}(y, \psi) \tag{7}$$

in the infinite domain $-\infty < y < \infty$ will generate a periodic solution of equation (3) with the *same eigenvalue*. In effect the transformation (6) replaces the periodic problem in real space (ψ, θ) with an equivalent problem (with the same operator and the same eigenvalue) in the infinite domain. The point of this crucial step is that $\hat{\xi}(y, \psi)$ *can* now be represented in the eikonal form $\hat{\xi} = A \exp[\mathrm{i}nS(\zeta, y, \psi)]$, where

$$S(\zeta, y, \psi) = \zeta - q(\psi)\, y + \int^{\psi} \mathrm{d}\psi q' y_0(\psi) \tag{8}$$

and $A(y, \psi)$ is a slowly varying amplitude which can be calculated by expansion in n^{-1}.

When this is done the m.h.d. eigenvalue equation becomes, in leading order in n^{-1},

$$\boldsymbol{B}\cdot\nabla\left\{\frac{|\nabla S|^2}{B^2}\,\boldsymbol{B}\cdot\nabla A\right\} + \rho\,\frac{\omega^2}{B^2}\,|\nabla S|^2 A - \frac{2p'\boldsymbol{\kappa}\wedge\boldsymbol{B}\cdot\nabla S}{B^2}\,A = 0, \tag{9}$$

where $\boldsymbol{\kappa} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}$ is the curvature of a magnetic line of force ($\boldsymbol{b} \equiv \boldsymbol{B}/|B|$), ρ and $p(\psi)$ are the fluid density and pressure, and the eigenvalue ω^2 is assumed to be close to marginal stability (i.e. $\omega^2 R^2 q^2 \rho/p \ll 1$). Equation (9) is to be solved in the infinite y-domain in which equilibrium quantities such as $\boldsymbol{\kappa}$ and $\boldsymbol{B}$ are periodic, but ∇S contains secular terms proportional to the magnetic shear $q'(\psi)$ which therefore has a marked influence on ω and hence on stability. The boundary condition to be satisfied by A is determined by the requirement that the transformation (11) converge, and the solution $A(y)$ then determines the *local* two-dimensional structure of $\xi(\theta, \psi)$. Equation (9) also determines a local eigenvalue $\hat{\omega}^2(\psi, y_0(\psi))$ which depends parametrically on ψ and the arbitrary poloidal angle y_0. To determine the relation between this intermediate eigenvalue $\hat{\omega}^2$ and the global eigenvalue ω^2 higher-order theory in n^{-1} is required. When this higher-order theory is developed (Connor *et al.* 1979*a*, *b*) one finds first a condition determining the unknown y_0; it must be at a minimum of $\hat{\omega}^2(y_0)$ which, because of symmetry, is frequently on the horizontal mid-plane at $y_0 = 0$. Then with y_0 fixed one finds a second ordinary differential equation for the amplitude A, this time in the flux coordinate ψ:

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}\frac{\partial^2\hat{\omega}^2}{\partial y_0^2}+(q')^2\left[2n(\omega^2-\omega_0^2)-x^2\frac{\partial^2\hat{\omega}^2}{\partial\psi^2}\right]A=0, \tag{10}$$

where $\omega_0^2 \equiv \hat{\omega}^2(\psi_0, y_0)$ is the minimum value of $\hat{\omega}^2$ and $x = (\psi-\psi_0)\,n^{\frac{1}{2}}$. This gives a Gaussian solution for $A(x)$ and determines the global eigenvalue ω^2 as

$$\omega^2 = \hat{\omega}^2(\psi_0, y_0)+\frac{1}{2n|q'|}\left(\frac{\partial^2\hat{\omega}^2}{\partial y_0^2}\,\frac{\partial^2\hat{\omega}^2}{\partial\psi^2}\right)^{\frac{1}{2}} \tag{11}$$

showing that the shortest-wavelength modes, $n \rightarrow \infty$, are the most unstable.

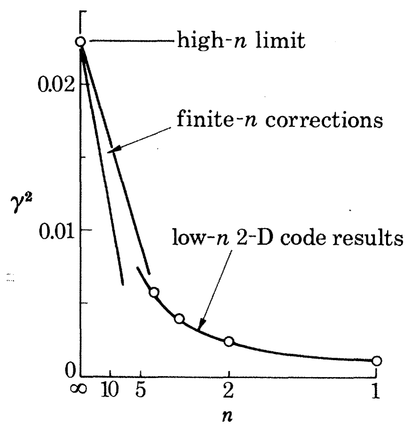

FIGURE 2. Growth rate for an unstable equilibrium in JET, plotted against toroidal mode number n. This shows $n \rightarrow \infty$ limit, finite-n corrections, and low-n ($n = 1, \ldots, 4$) results obtained from a two-dimensional computer code.

Thus it can be seen that the present technique allows one to break down the original two-dimensional eigenvalue problem into two successive one-dimensional problems. The first, in the extended poloidal variable y, determines a local eigenvalue $\hat{\omega}^2(\psi, y_0)$ and determines the local two-dimensional structure of the mode. The local eigenvalue $\hat{\omega}^2$ then provides the coefficients for a second one-dimensional equation which determines the global eigenvalue ω^2 and the slow transverse dependence of the envelope of the mode $A((\psi-\psi_0)\,n^{\frac{1}{2}})$. An example of the application of these methods is shown in figure 2 in which growth rate is plotted against n for an unstable equilibrium in JET. The figure also shows that the growth rates calculated by the present technique for large mode numbers n connect smoothly onto those computed by the two-dimensional, Sykes–Wesson, initial-value code for low mode numbers.

(c) *Alternative representations of toroidal eigenmodes*

Before we consider applications of the transformation (6) to other plasma models and stability problems it is instructive to consider other versions of this representation. Several independent groups (Pegoraro & Schep 1978, Lee & Van Dam 1979, Glasser 1979) developed, almost simultaneously, equivalent representations for toroidal eigenmodes but alternative approaches offer different insights. Lee & Van Dam (1979), for example, argued along the following lines. For each value of n, a toroidal eigenmode will be composed of many poloidal harmonics $\xi_m e^{im\theta}$. Because of the requirement of long parallel wavelength each poloidal harmonic will be centred on (and localized about) its own mode rational surface, i.e. that magnetic surface on which $\boldsymbol{k}_m \cdot \boldsymbol{B} \propto (m - nq(\psi)) = 0$. Neighbouring ξ_m must therefore be separated by a small flux $\Delta\psi = 1/nq'$ where, if $m = nq(\psi)$, then $m+1 = nq(\psi+\Delta\psi)$. Now, since equilibrium quantities do not vary on this scale there must be *local translational invariance* and the ξ_m will be related by

$$\xi_m(\psi) = e^{i\lambda}\xi_{m-1}(\psi-\Delta\psi) = \dots = e^{im\lambda}\xi(\psi - m\Delta\psi). \qquad (12)$$

Thus, remarkably, the mathematical treatment of short-wavelength toroidal eigenmodes is similar to the Bloch analysis of lattices in solid state physics. Using equation (12) and the Poisson sum formula Lee & Van Dam arrived at the following series representation for $\xi(\theta, \psi)$:

$$\xi(\theta, \psi) = \sum_{l=-\infty}^{\infty} F(\theta+\lambda+2\pi l) \exp\left[inq(\theta+\lambda+2\pi l)\right] \qquad (13)$$

which is similar to the form developed at Princeton by Glasser and others, and equivalent to (6) plus the eikonal transformation. Other authors (notably, Pegoraro & Schep 1978; Tsang & Catto 1977) followed a similar line of argument.

What is clearly revealed in this approach is the true nature of the n^{-1} expansion. This is an expansion in $\Delta\psi/\psi \equiv (n\psi q')^{-1}$. Evidently for low shear equilibria, $\psi q'/q \ll 1$, higher values of n are required for the validity of the theory, and agreement between computed, low-n, results and high-n theory may be more difficult to obtain.

In an alternative treatment of equation (9) this equation is solved for $A(y)$ with $y_0(\psi, \omega)$ as eigenvalue. A higher-order theory (Chance *et al.* 1979) in n^{-1} then generates a second eigenvalue equation in the form of a W.K.B. phase integral of the form

$$n|q'| \oint [y_0(\psi, \omega)]^{\frac{1}{2}} \, d\psi = (N+\tfrac{1}{2})\,\pi \qquad (14)$$

which determines the global eigenvalue ω. Within ideal m.h.d. the most unstable mode is $N = 0$, and this phase integral gives the same eigenvalue as equation (10), but in other stability problems it represents a significant generalization of the higher-order theory.

4. Short-wavelength resistive modes in a torus

When the new mode representation is applied to the resistive m.h.d. equations in a torus the eigenvalue problem reduces, in leading order, to a pair of coupled one-dimensional equations (Glasser 1979, Chance *et al.* 1979). By taking the low-frequency limit again, for simplicity, these reduce to a single resistive eigenmode equation (Bateman & Nelson 1978):

$$\boldsymbol{B}\cdot\nabla \left\{\frac{|\nabla S|^2}{B^2(1+(i\eta/\omega)\, n^2|\nabla S|^2)}\right\} \boldsymbol{B}\cdot\nabla A + \frac{\omega^2\rho|\nabla S|^2}{B^2} A - \frac{2p'\boldsymbol{\kappa}\wedge\boldsymbol{B}\cdot\nabla S}{B^2} A = 0, \qquad (15)$$

which differs from its ideal m.h.d. counterpart (9) only in the appearance of a 'small' secular resistive term proportional to $n^2\tau_A/\tau_r$, where $\tau_A = Rq(\rho/B^2)^{\frac{1}{2}}$ is the Alfven transit time and $\tau_r = a^2/\eta c^2$ is the resistive diffusion time. Because $\tau_A/\tau_r < 10^{-6}$ in a typical tokamak, for moderate values of n, $n^2\tau_A/\tau_r \ll 1$ and the effect of resistivity is to modify the ideal m.h.d. equation only at large values of the independent variable y. Resistivity therefore has the effect of modifying the boundary conditions as $y \to \pm\infty$, on the ideal m.h.d. equation. This demonstrates the Fourier-transform nature of the extended poloidal variable y, since in real space the effect of resistivity is important only in a narrow layer around $\boldsymbol{k}\cdot\boldsymbol{B} = 0$.

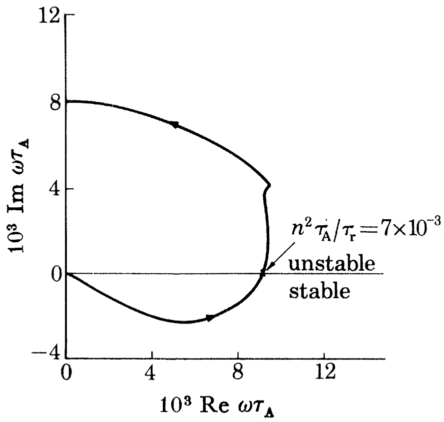

FIGURE 3. Behaviour of the complex frequency ω as a function of $n^2\tau_A/\tau_r$ for resistive m.h.d. modes. The arrows are in the sense of increasing $n^2\tau_A/\tau_r$.

The resistive equations have been investigated in detail in a series of elegant papers by the Princeton group (see, for example, Glasser *et al.* 1979). Figure 3 shows an example of the complex mode frequency as a function of $n^2\tau_A/\tau_r$. This shows stability for $n^2\tau_A/\tau_r < 7\times10^{-3}$, and instability for higher values of n.

Thus detailed calculations of the stability of both ideal and resistive modes within the m.h.d. model can now be made for any axisymmetric toroidal equilibrium. Such calculations show that two dissimilar ideal m.h.d. instabilities are the most persistent and they place the severest restrictions on β. These are (i) the low-n free-boundary modes (see, for example, Chance *et al.* 1979) occurring when a vacuum region separates the plasma from a conducting wall, and (ii) the $n \to \infty$ internal modes described by the new theory. They also show, as predicted by Roberts & Taylor (1965), that as $n \to \infty$ a resistive pressure-driven mode always becomes unstable. The m.h.d. model breaks down however as $n \to \infty$ since perpendicular wavelengths become comparable to the ion Larmor radius. To resolve this it is necessary to turn to the more detailed description afforded by the Vlasov or Fokker–Planck equations which also permit the investigation of a variety of micro-instabilities such as the electron drift wave.

5. Vlasov–Maxwell stability

The transformation (6), or one of its equivalent forms, can be used directly on the perturbed distribution functions and fields in the Vlasov or Fokker–Planck equations, though some authors delay its introduction until approximate solutions of these equations have been obtained in real space. To retain finite Larmor radius (f.L.r.) effects the mode number n is related to the small Larmor radius parameter so that $na_i/L \sim O(1)$, where a_i is the ion Larmor radius and L is an equilibrium scale length. In lowest order in n^{-1} the two-dimensional eigenvalue problem in ψ, θ is reduced to a set of three integro-differential equations (Lee & Van Dam 1979; Antonsen & Lane 1980; Tang *et al.* 1980). These equations contain complicated kernels involving multi-dimensional velocity–space integrations, but the structure of the problem is once again *one-dimensional* in the extended poloidal variable $y(-\infty < y < \infty)$, with parametric dependence on ψ and $y_0(\psi)$. The equations contain many kinetic effects not described by fluid theory, and are currently the centre of considerable theoretical activity.

For a low-β plasma ($\beta \ll a/Rq^2$) electron drift waves become predominantly electrostatic and a single eigenmode equation has been derived for such waves. Computer codes for the solution of this integral equation have been written and solutions obtained (Frieman *et al.* 1979; Connor *et al.* 1979*c*).

The fully electromagnetic equations have been derived in both the collisionless limit and in a collisional limit and, by specializing to magnetohydrodynamic modes, generalizations of the m.h.d. eigenvalue equations (9) or (15) have been obtained. Thus, including collisions, Lee *et al.* (1979) have obtained a kinetic generalization of the resistive m.h.d. equation:

$$\boldsymbol{B}\cdot\nabla\left\{\frac{|\nabla S|^2}{B^2(1+\hat{\eta}n^2|\nabla S|^2/\omega)}\,\boldsymbol{B}\cdot\nabla A\right\}+\omega(\omega-\omega_{*\mathrm{i}})\,\frac{|\nabla S|^2\rho}{B^2}\,A-2p'\,\frac{\boldsymbol{\kappa}\times\boldsymbol{B}\cdot\nabla S}{B^2}\,A=0, \tag{16}$$

where $\hat{\eta}$ is a modified resistivity, $\hat{\eta} = \eta(1-\omega_{*\mathrm{e}}/\omega)$, $\omega_{*\mathrm{i}} = (nT_\mathrm{i}/e_\mathrm{i})\,(\partial\,(\ln p_\mathrm{i})/\partial\psi)$ and

$$\omega_{*\mathrm{e}} = n(T_\mathrm{e}/e_\mathrm{e})\,(\mathrm{d}\,(\ln\rho)/\mathrm{d}\psi).$$

Because of the coupling of the n^{-1} expansion to the a_i/L-expansion the higher-order theory in the Vlasov–Maxwell model is extremely complicated, but Frieman *et al.* (1979) have given the appropriate version for an electrostatic mode in slab geometry and, as conjectured by many other authors, the W.K.B. phase integral condition (14) emerges as the eigenvalue equation.

Summary and conclusions

A novel method of representing short-wavelength eigenmodes in a torus has been developed recently. This has led to a complete theory of short-wavelength m.h.d. modes, complementing the existing numerical methods for determining the stability of long-wavelength modes in a torus.

The m.h.d. stability of any toroidal equilibrium can now be precisely determined.

The new methods are also applicable to the more basic set of plasma equations – the Vlasov or Fokker–Planck equations, for ions and electrons, coupled to the Maxwell equations. These equations permit the investigation of kinetic effects on m.h.d. modes as well as of various micro-instabilities. Thus the recent advances described here have also opened up new possibilities of accurately calculating the linear stability of high-temperature, almost collisionless plasmas whose behaviour is not adequately described by fluid equations.

The Culham Laboratory contribution to the work described above is due largely to Dr J. B. Taylor, F.R.S., and Dr J. W. Connor with whom I have had the pleasure of working and discussing much of this material. Several other groups of authors have developed similar ideas

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

to which I have not been able to do full justice, but I should like to acknowledge my indebtedness to Dr A. H. Glasser, Dr R. L. Dewar, Dr F. Pegoraro and Dr T. J. Schep for illuminating discussions and to Dr Y. C. Lee and Dr J. W. Van Dam whose work I have studied with interest.

References (Hastie)

Antonsen, T. M. Jnr. & Lane, B. 1980 *Physics Fluids* **23**, 1205–1214.
Bateman, G. & Nelson, D. B. 1978 *Phys. Rev. Lett.* **41**, 1804–1807.
Berger, D., Bernard, L. C., Gruber, R. & Troyon, F. 1977 *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 6th Int. Conf., Berchtesgaden, 1976), vol. 2, pp. 411–421. Vienna: I.A.E.A.
Bussac, M. N., Pellat, R., Edery, D., Soulé, J. L. 1975 *Phys. Rev. Lett.* **35**, 1638–1641.
Chance, M. S., Dewar, R. L., Frieman, E. A., Glasser, A. H., Greene, J. M., Grimm, R. C., Jardin, S. C., Johnson, J. L., Manickam, J., Okabayashi, M. & Todd, A. M. M. 1979 *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 7th Int. Conf., Innsbruck, 1978), vol. 1, pp. 677–687. Vienna: I.A.E.A.
Connor, J. W., Hastie, R. J. & Taylor, J. B. 1978 *Phys. Rev. Lett.* **40**, 396–399.
Connor, J. W., Hastie, R. J. & Taylor, J. B. 1979*a* *Proc. R. Soc. Lond.* A **365**, 1–17.
Connor, J. W., Hastie, R. J. & Taylor, J. B. 1979*b* *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 7th Int. Conf., Innsbruck, 1978), vol. 1, pp. 667–675. Vienna: I.A.E.A.
Connor, J. W., Hastie, R. J., Hesketh, K. W. & Taylor, J. B. 1979*c* *Controlled Fusion and Plasma Physics* (Proc. 9th European Conf., Oxford, 1979), p. 10.
Coppi, B., Greene, J. M. & Johnson, J. L. 1966 *Nucl. Fus.* **6**, 101–117.
Frieman, E. A., Greene, J. M., Johnson, J. L. & Weimer, K. E. 1973 *Physics Fluids.* **16**, 1108–1125.
Frieman, E. A., Rewoldt, G., Tang, W. M. & Glasser, A. H. 1979 *Princeton Plasma Phys. Lab. Rep.* PPPL 1560.
Furth, H. P., Killeen, J. & Rosenbluth, M. N. 1963 *Physics Fluids.* **6**, 459–484.
Glasser, A. H. 1979 *Proceedings of the Finite Beta Workshop, Varenna 1977*, U.S. Department of Energy Conf. 7709167, p. 55.
Glasser, A. H., Chance, M. S. & Dewar, R. L. 1979 *Controlled Fusion and Plasma Physics* (Proc. 9th European Conf., Oxford, 1979), p. 8.
Hamada, S. 1958 *Kaku-Yugo Kenkyu* **1**, 542–563.
Johnson, J. L., Chance, M. S., Greene, J. M., Grimm, R. C., Jardin, S. C., Kerner, W., Manickam, J. & Weimer, K. E. 1977 *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 6th Int. Conf., Berchtesgaden 1976), vol. 2, pp. 395–400. Vienna: I.A.E.A.
Lee, Y. C. & Van Dam, J. W. 1979 *Proceedings of the Finite Beta Workshop, Varenna 1977*, U.S. Department of Energy Conf. 7709167, p. 93.
Lee, Y. C., Van Dam, Drake, J. F., Lin, A. T., Pritchett, P. L., D'Ippolito, D., Liewer, P. C. & Liu, C. S. 1979 *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 7th Int. Conf., Innsbruck, 1978), vol. 1, pp. 799–807. Vienna: I.A.E.A.
Mercier, C. 1960 *Nucl. Fus.* **1**, 47–53.
Newcomb, W. A. 1960 *Ann. Phys.* **10**, 232–267.
Pegoraro, F. & Schep, T. J. 1978 *Rijnhuizen Rep.* 78–116, FOM-Institut voor Plasma fysica, Rijnhuizen, Jutphaas, Netherlands.
Roberts, K. V. & Taylor, J. B. 1965 *Physics Fluids.* **8**, 315–322.
Schneider, W. & Bateman, G. 1975 *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 5th Int. Conf., Tokyo, 1974), vol. 1, pp. 429–438. Vienna: I.A.E.A.
Shafranov, V. D. 1970 *Zh. tekh. Fiz.* **40**, 241–253 (transl. as *Soviet Phys. tech. Phys.* **15**, 175–183 (1971)).
Sykes, A. & Wesson, J. A. 1974 *Nucl. Fus.* **14**, 645–648.
Tang, W. M., Connor, J. W. & Hastie, R. J. 1980 *Nucl. Fus.* (To be published.)
Tsang, K. T. & Catto, P. J. 1977 *Phys. Rev. Lett.* **39**, 1664–1667.
Ware, A. A. & Haas, F. A. 1966 *Physics Fluids.* **9**, 956–964.
Wesson, J. A. 1978 *Nucl. Fus.* **18**, 87–132.